PARABOLIC PDEs AND RICCI FLOW

Syafiq Johar
syafiq.johar@maths.ox.ac.uk

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1 Introduction

The Ricci flow is an evolution equation of the metric on a Riemannian manifold introduced by Richard Hamilton in 1982. The heat-type evolution equation is given by

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) & (1) \\ g(0) = g_0 & (2) \end{cases}$$

where Ric is the Ricci curvature of the metric $g(t)$ at time t. The Ricci flow is considered a "heat-type" evolution equation because in harmonic coordinates, we can show that the Ricci curvature can be written as

$$\mathrm{Ric}(g)_{ij} = \frac{1}{2}\Delta g_{ij} + Q_{ij}(g^{-1}, \partial g) \tag{3}$$

where Δ is the Laplace-Beltrami operator and $Q(g^{-1}, \partial g)$ a quadratic form involving the inverse of the metric g and its derivatives. In this coordinates, the Ricci flow is a non-linear heat equation (but it is not strongly parabolic as we shall see later).

Intuitively, as a heat-type equation, it will try to diffuse the metric over the Riemannian manifold. Therefore, it will aim to distribute the metric (and hence curvature) evenly throughout the manifold over time. Therefore, we expect the manifold will become more and more homogeneous. In particular, if the manifold is simply connected and compact, we expect it to get rounder and rounder as time progresses.

However, due to the local nature of the non-linear heat type equation, global solutions is generally not possible. We can only expect the existence of solutions in a short interval of time. Therefore, singularities might occur at some finite time.

We also have the problem of proving the existence of the solution to the Ricci flow. Since the Ricci flow is only weakly parabolic, we do not have the usual machineries for the theory of parabolic partial differential equations. However, Hamilton was able to show existence and uniqueness by using the Nash-Moser implicit function theorem. Later, DeTurck gave an easier proof which is today known as DeTurck's trick.

2 Parabolic PDEs on Scalar Functions

2.1 Symbol and Principal Symbol of Linear PDEs

A linear partial differential operator in the Euclidean space $\mathbb{R}^n$ of order k is the operator $P : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ such that for any sufficiently smooth u, we have

$$Pu = \sum_{|\alpha| \leq k} a^\alpha \partial^\alpha u$$

where α is a multi-index and $a^\alpha : \mathbb{R}^n \to \mathbb{R}$. Sometimes, we can write it simply as

$$P(x, \partial) = \sum_{|\alpha| \leq k} a^\alpha \partial^\alpha.$$

The symbol of a differential operator is obtained by replacing each partial derivative with a distinct variable. So, loosely speaking, the symbol of a differential operator of order k is a degree k polynomial.

Definition 2.1. The symbol $\sigma_P(x, \xi)$ of of the partial differential operator P at the point x is given by the polynomial over n variables $\xi = (\xi_1, \ldots, \xi_n)$ as

$$\sigma_P(x, \xi) = \sum_{|\alpha \leq k} a^\alpha \xi_\alpha.$$

Most of the time, we are interested with the leading or principal symbol $\hat{\sigma}_P(x, \xi)$ of the operator P, which is the leading term in the polynomial $\sigma_P(x, \xi)$

$$\hat{\sigma}_P(x, \xi) = \sum_{|\alpha| = k} a^\alpha \xi_\alpha.$$

This generalises to a linear partial differential operator on Riemannian manifolds. Given a linear partial differential operator P on a manifold (M, g), the principal symbol $\hat{\sigma}_P : T^*M \to \mathbb{R}$ is a function on the cotangent bundle given by

$$\hat{\sigma}_P(x, \xi) = \sum_{|\alpha| = k} a^\alpha \xi_\alpha$$

where (x, ξ) is a local trivialisation of the cotangent bundle. This is well defined since it is independent of local coordinates.

The symbol of an operator is widely used in Fourier analysis (recall that Fourier transform maps derivatives to multiplication with polynomials). Therefore, it is common in literature to have $i\xi$ instead of just ξ in the definition for the symbol of an operator for this purpose. However, defining it without the imaginary number i is useful for defining parabolicity of a partial differential equation, which we are going to do in the next subsection.

2.2 Linear Second Order Parabolic PDEs

Let $\Omega \subset \mathbb{R}^n$ be an open subset of the Euclidean space $\mathbb{R}^n$ for $n \geq 2$ and $T > 0$. We define the space $Q_T = \Omega \times (0, T)$ as the parabolic cylinder. Consider the following second order linear partial differential equation in the domain Q_T

$$\frac{\partial u}{\partial t} = a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + cu \tag{4}$$

for smooth coefficients $a^{ij}, b^i, c : Q_T \to \mathbb{R}$.

Definition 2.2 (Parabolic and uniformly parabolic PDE)**.** We say that the equation is (strongly) parabolic if the matrix $(a^{ij}(x,t))$ is positive definite everywhere in the domain Q_T i.e. there exists a positive function $\lambda : Q_T \to \mathbb{R}_{>0}$ such that

$$a^{ij}\xi_i\xi_j \geq \lambda(x)|\xi|^2 \tag{5}$$

for all $\xi \in \mathbb{R}^n$. The equation is called (strongly) uniformly parabolic if the matrix $(a^{ij}(x,t))$ is uniformly positive definite i.e. there exists a constant $\lambda > 0$ such that

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \tag{6}$$

for all $\xi \in \mathbb{R}^n$.

However, as we shall work on compact domains/manifolds most of the time, we shall see the uniformly parabolic definition used more regularly.

Remark 2.1. Note that we include the term strongly in parenthesis. This is because there exists a notion of weakly parabolic equations. Instead of being positive definite, we allow the matrix $(a^{ij}(x,t))$ to be positive semi-definite i.e. the function $\lambda(x,t)$ is non-negative instead of positive. This is an important distinction because most results on parabolic equation only holds for the strongly parabolic type. From now on, we refer strongly parabolic equations simply by parabolic equations and make the distinction for weakly parabolic equations where necessary.

An obvious and simple example of a uniformly parabolic equation in the Euclidean space $\mathbb{R}^n$ for some n is the heat equation, given by the PDE

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \tag{7}$$

In fact, since the equation only deals with local terms, we can define such equation on Riemannian manifolds. Most of the existence results for PDEs on $\Omega \subset \mathbb{R}^n$ requires the definition of norms on spaces of functions. To extend this to manifolds M, we just need to redefine the norms by introducing charts locally and partitions of unity globally.

Let (M, g) be a closed Riemannian manifold and $u : M \times [0, T] \to \mathbb{R}$ is a function on the manifold in some time interval $[0, T]$, then consider the equation

$$\frac{\partial u}{\partial t} = L(u) \tag{8}$$

where $L : C^\infty(M) \to C^\infty(M)$ is a second order partial differential operator given by:

$$L(u) = -\Delta_g u + \langle X(t), \nabla u\rangle + c(x, t)u \tag{9}$$

where Δ_g is the Laplace-Beltrami operator with respect to the metric $g(t)$ on M, $X(t)$ is a time varying smooth vector field on M and $c(x, t)$ is a smooth function on $M \times [0, T]$. In local coordinates $\{x^i\}$ of M, the operator L can be written as

$$L(u) = a^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + cu$$

for smooth coefficients a^{ij}, b^i, c such that $(a^{ij}(x, t))$ is positive definite for all $x \in M$ in the sense of (5), then the equation is called parabolic (and uniformly parabolic if it is positive definite in the sense of (6)). This is well defined as it is independent of choice of coordinates.

We can also define parabolicity using the principal symbol of the partial differential operator, which we are going to do next:

Definition 2.3. The partial differential equation in (8) on a Riemannian manifold (M, g) is parabolic if the principal symbol of L is positive i.e. $\hat{\sigma}_L(x, \xi) > 0$ for all trivialisations $(x, \xi) \in T^*M$ with $\xi \neq 0$.

Remark 2.2. As before, we distinguish it with weakly parabolic equations in which we allow $\hat{\sigma}_L(x, \xi) = 0$ for some $x \in M$ and $\xi \neq 0$.

Example 2.1 (Laplace-Beltrami operator and heat equation)**.** Consider the Laplace-Beltrami operator Δ_g on functions $u : M \to \mathbb{R}$. This is given by the negative of the composition of the div and grad operator, which can be written in local coordinates as

$$\begin{aligned}\Delta_g u = -\mathrm{div}(\mathrm{grad}(u)) &= -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^j}\left(\sqrt{|g|}g^{ij}\frac{\partial u}{\partial x^i}\right) \\ &= -g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + \text{lower order derivatives on } u\end{aligned}$$

where $|g| = |\det(g)|$. Thus, the principal symbol for the operator $-\Delta_g$ is

$$\hat{\sigma}_{-\Delta_g}(x, \xi) = g^{ij}\xi_i\xi_j = |\xi|^2 > 0$$

for any nonzero ξ. Therefore, the heat equation on a manifold (M, g) given by

$$\frac{\partial u}{\partial t} = -\Delta_g u \tag{10}$$

is a parabolic equation.

Remark 2.3. One needs to be careful with the Laplace-Beltrami operator. The sign convention differs from author to author. We used this sign convention because it would ensure that the Laplace-Beltrami operator and the Hodge Laplacian coincide (i.e. $\Delta_g = dd^* + d^*d$) for functions. A simple way to define the Laplace-Beltrami operator is to require Green's theorem holds i.e. for all smooth compactly supported function $\phi \in C_c^\infty(M)$, we have:

$$\int_M g(\nabla u, \nabla f)\, dV_g = \int_M (\Delta u)\phi\, dV_g$$

where dV_g is the volume element of the Riemannian manifold (M, g).

Another notion of Laplacian in geometric analysis is the trace Laplacian which is given by the trace of second covariant derivatives on k-forms. It can be shown that if u is a function (or 0-form), the trace Laplacian, given by $\tilde{\Delta} u = -\mathrm{tr}(\nabla^2 u)$, coincides with the Laplace-Beltrami operator. However, on higher order forms, they differ by a curvature form and can be related by the Weitzenböck formula.

2.2.1 Some Results on Linear Parabolic PDEs

Parabolic linear equations have been widely studied and has lots of useful results, particularly in regularity, existence and uniqueness.

Suppose that (M, g) is a compact n-dimensional Riemannian manifold. Suppose that we have the following homogeneous linear parabolic equation, which can be written locally as:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^{ij}\dfrac{\partial^2 u}{\partial x^i \partial x^j} + b^i \dfrac{\partial u}{\partial x^i} + cu \quad \text{in} \quad M \times (0, T] \\ u(M, 0) = 0 \end{cases} \tag{11}$$

where $a^{ij}, b^i, c : M \times [0, T] \to \mathbb{R}$ are smooth coefficients and (a^{ij}) symmetric and positive definite.

Theorem 2.1. [Aub, p.131] There exists a unique and smooth solution $u \in C^\infty(M \times [0, T])$ satisfying (11).

In fact, this can be extended to the inhomogeneous linear parabolic equation, which can be written locally as:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^{ij}\dfrac{\partial^2 u}{\partial x^i \partial x^j} + b^i \dfrac{\partial u}{\partial x^i} + cu + f \quad \text{in} \quad M \times (0,T] \\ u(M,0) = 0 \end{cases} \tag{12}$$

where f is a function on $M \times [0,T]$. Choosing $p > n+2$, we have a generalisation:

Theorem 2.2. [Aub, p.131] For every $f \in L^p(M \times [0,T])$, there exists a unique solution $u \in H^p(M \times [0,T])$ satisfying (12).

Furthermore, these two results above have also been proven for Riemannian manifolds with boundary by Hamilton [Ham, p.120]. Another important result in the theory of linear parabolic equations is the maximum principle, which is often used to obtain bounds on the evolving quantities.

Theorem 2.3 (Weak maximum principle)**.** [Eva, p.390] Suppose that (M,g) is a compact Riemannian manifold with boundary ∂M and $u \in C^\infty(M \times (0,T])$ satisfies the parabolic inequality, which can be written locally as:

$$\frac{\partial u}{\partial t} \le a^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} \quad \text{in} \quad M \times (0,T] \tag{13}$$

where $a^{ij}, b^i : M \times [0,T] \to \mathbb{R}$ are smooth coefficients and (a^{ij}) symmetric and positive definite. Then

$$\max_{M\times[0,T]} u = \max_{\Gamma_T} u \tag{14}$$

where $\Gamma_T = (\partial M \times [0,T]) \cup (M \times \{0\})$.

Remark 2.4. We also have the weak minimum principle where if we replace the $\le$ in (13) with $\ge$, the max on both sides of (14) is then replaced with min.

We also have the strong maximum principle, which gives us a stronger consequence of the parabolic inequality.

Theorem 2.4 (Strong maximum principle)**.** [Eva, p.397] Suppose that (M,g) is a connected Riemannian manifold with boundary ∂M and $u \in C^\infty(M \times (0,T])$ satisfies the parabolic inequality, which can be written locally as:

$$\frac{\partial u}{\partial t} \le a^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + cu \quad \text{in} \quad M \times (0,T] \tag{15}$$

where $a^{ij}, b^i, c : M \times [0,T] \to \mathbb{R}$ are smooth coefficients, $c \le 0$, (a^{ij}) is symmetric and positive definite. Then, if u attains a non-negative maximum over $M \times [0,T]$ at a point $(x_0,t_0) \in (M \setminus \partial M) \times (0,T]$, then u is constant on $(M \setminus \partial M) \times (0,t_0]$.

Remark 2.5. Again, we also have the strong minimum principle where if we replace the $\leq$ in (15) with $\geq$ and assume that u attains a non-positive minimum over $M \times [0,T]$ at a point $(x_0, t_0) \in (M \setminus \partial M) \times (0,T]$, then u is constant on $(M \setminus \partial M) \times (0, t_0]$.

In fact, we can refine these results on closed manifolds to get an explicit bound for replacing the term cu with some $f(u,t)$ where $F : \mathbb{R} \times [0,T] \to \mathbb{R}$ is a smooth function (we may now think of cu as $f(u) = cu$, where f is a smooth function of u it is just a multiplication with a smooth function c). We get the following result:

Theorem 2.5. [Top, p.35] Suppose that (M,g) is a closed Riemannian manifold and $u \in C^\infty(M \times (0,T])$ is a subsolution to the parabolic equation, which can be written locally as:

$$\begin{cases} \dfrac{\partial u}{\partial t} \leq a^{ij} \dfrac{\partial^2 u}{\partial x^i \partial x^j} + b^i \dfrac{\partial u}{\partial x^i} + f(u,t) \quad \text{in} \quad M \times (0,T] \\ u(M,0) \leq C \in \mathbb{R} \end{cases} \tag{16}$$

where $a^{ij}, b^i : M \times [0,T] \to \mathbb{R}$ are smooth coefficients, (a^{ij}) is symmetric and positive definite and $F : \mathbb{R} \times [0,T] \to \mathbb{R}$ is a smooth function. Suppose further that $\phi : [0,T] \to \mathbb{R}$ solves the following ODE:

$$\begin{cases} \dfrac{d\phi}{dt} = f(\phi(t), t) \\ \phi(0) = C. \end{cases} \tag{17}$$

Then, for all $t \in [0,T]$, we have

$$u(M,t) \leq \phi(t) \tag{18}$$

Remark 2.6. Similar as before, we can have a minimum principle result for supersolutions where if we replace all the $\leq$ in (16) with $\geq$, then the $\leq$ in (18) is also replaced with $\geq$.

Remark 2.7. An equivalent result to the strong maximum principle for Theorem 2.5 is that we must have $u(M,t) < \phi(t)$ for all $t \in (0,T]$ unless $u(M,t) = \phi(t)$ for all $t \in [0,T]$.

2.3 Non-Linear Second Order Parabolic PDEs

We have seen the linear second order parabolic PDEs and the nice related results. However, linear parabolic PDEs are not common: most of the time, we have to deal with non-linear PDEs, which are much harder to deal with. In local coordinates, one frequently have to deal with differential equations of some form involving the term

$$\tilde{F}(x,u) := F(x,u,\partial u, \dots \partial^k u) = \sum_{|\alpha|=k} a^\alpha(x,u,\partial u, \dots, \partial^{k-1} u)\partial^\alpha u + f(x,u,\partial u, \dots, \partial^{k-1} u)$$

where f is a smooth function on its variables. This PDE is called the quasilinear PDE in which it is linear in the highest derivative of u. If for all $|\alpha| = k$, all a^α depend only on x, the equation is called a semilinear PDE. If additionally a^α depend on ∂u^k, then the equation is called a fully non-linear PDE.

All the theorems we saw in the previous subsection holds only for linear PDEs. However, we can still make sense of local behaviour of the function u, which evolves non-linearly, by linearising it at u.

Definition 2.4 (Linearisation)**.** For a non-linear differential operator $\tilde{F}(x, u)$, its linearisation or first variation at u is the linear operator L such that for sufficiently smooth v, we have

$$L(v) = (D\tilde{F})(v) = \frac{d}{d\varepsilon}\tilde{F}(x, u + \varepsilon v)|_{\varepsilon=0}$$

Example 2.2. Suppose we have the quasilinear second order differential operator on $\mathbb{R}^2$ given by $\tilde{F}(x, y, u) = u\partial_y^2 u + y\partial_x^2 u$. The linearisation at u is given by:

$$L(v) = u\partial_y^2 v + y\partial_x^2 v + v\partial_y^2 u. \tag{19}$$

Clearly, this is linear in v.

The linear second order parabolic equation in (8) generalises to the non-linear case of a more general form

$$\frac{\partial u}{\partial t} = F(x, t, u, \partial_x u, \partial_x^2 u) \tag{20}$$

where F is a smooth function of its variables. To define the notion of parabolicity, we look at the linearisation of the function F.

Definition 2.5 (Non-linear second order parabolic PDE)**.** Equation (20) is called a parabolic non-linear equation at u if the linear equation involving the linearisation of F at u i.e. Lv for sufficiently smooth v, given by

$$\frac{\partial v}{\partial t} = L(v)$$

is parabolic in the sense of Definition 2.2.

Example 2.3. From the previous example, the quasilinear equation $\frac{\partial u}{\partial t} = u\partial_y^2 u + y\partial_x^2 u$ is parabolic at the points (x, y) where there exists a positive function $\lambda(x, y) > 0$ such that for any $(\xi, \zeta) \neq (0, 0)$, we have $\xi^2 y + \zeta^2 u \geq \lambda(x, y)(\xi^2 + \zeta^2)$.

2.3.1 Some Results on Quasilinear Parabolic PDEs

Most of the time in the study of Ricci flows, we are interested in quasilinear parabolic equations only where in equation (20), the smooth function F has the form:

$$F(x,t,u,\partial_x u,\partial_x^2 u) = a^{ij}(x,t,u,\partial_x u)\frac{\partial^2 u}{\partial x^i \partial x^j} + f(x,t,u,\partial_x u) \tag{21}$$

where f is a smooth function on its variables. The linearisation of a quasilinear parabolic equation preserves the leading term as it is already linear in the highest order.

Unlike linear parabolic PDEs, non-linear parabolic PDEs may not guarantee long time existence. Consider the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = F(x,t,u,\partial_x u,\partial_x^2 u) \\ u(M,0) = u_0(x) \end{cases} \tag{22}$$

where $F(x,t,u,\partial_x u,\partial_x^2 u)$ is given in (21) and $u_0(x)$ some initial condition. Using Schauder fixed point theorem, we have the following existence theorem:

Theorem 2.6 (Short time existence)**.** Suppose that $u_0 \in C^\infty(M)$, then there exists $\varepsilon > 0$ such that the problem (22) has a unique smooth solution $u(x,t)$ for $x(x,t) \in M \times [0,\varepsilon]$.

We also have the comparison theorem (similar to the strong maximum principle) by extending the linear case to the non-linear case. Let $F(x,t,u,\partial_x u,\partial_x^2 u) = F(x,t,p,q_i,r_{ij})$ in coordinates. Additionally, we require the matrix $(\frac{\partial F}{\partial r_{ij}})$ to be symmetric and positive definite. Let $u, v \in C^\infty(M)$ be solutions of (22), define $w = u - v$ and $z(x,t,s) = v + s(u-v)$ for $s \in [0,1]$. Then

$$\begin{aligned} F(x,t,u,\partial_x u,\partial_x^2 u) - F(x,t,v,\partial_x v,\partial_x^2 v) &= \int_0^1 \frac{\partial}{\partial s} F(x,t,z,\partial_x z,\partial_x^2 z)\, ds \\ &= \sum_{i,j=1}^n a^{ij}\frac{\partial^2 w}{\partial x^i \partial x^j} + \sum_{i=1}^n b^i \frac{\partial w}{\partial x^i} + cw \end{aligned}$$

where we have the following expressions for the coefficients:

$$\begin{aligned} a^{ij}(x,t) &= \int_0^1 \frac{\partial F}{\partial r_{ij}}(x,t,z,\partial_x z,\partial_x^2 z)\, ds \\ b^i(x,t) &= \int_0^1 \frac{\partial F}{\partial q_i}(x,t,z,\partial_x z,\partial_x^2 z)\, ds \\ c(x,t) &= \int_0^1 \frac{\partial F}{\partial p}(x,t,z,\partial_x z,\partial_x^2 z)\, ds \end{aligned}$$

such that a^{ij}, b^i, c are smooth and a^{ij} is symmetric positive definite. Note that this is a linear and elliptic equation for w. If $\frac{\partial F}{\partial p} = 0$ and $\frac{\partial F}{\partial p} \leq 0$ respectively (in other words,

$c = 0$ or $c \leq 0$ respectively), we can apply Theorem 2.4 to the parabolic equation for w given by:

$$\frac{\partial w}{\partial t} = F(x, t, u, \partial_x u, \partial_x^2 u) - F(x, t, v, \partial_x v, \partial_x^2 v).$$

Thus, we have the following comparison theorem:

Theorem 2.7 (Comparison theorem)**.** Suppose that (M, g) is a compact connected Riemannian manifold with boundary ∂M and $u, v \in C^\infty(M \times (0, T])$ satisfies the non-linear parabolic inequality (20) such that

$$F(x, t, u, \partial_x u, \partial_x^2 u) \geq F(x, t, v, \partial_x v, \partial_x^2 v)$$

with the additional conditions that the matrix $(\frac{\partial F}{\partial r_{ij}})$ is symmetric positive definite and $\frac{\partial F}{\partial p} \leq 0$. Then, if $u \leq v$ on $\Gamma_T = (\partial M \times [0, T]) \cup (M \times \{0\})$, then $u \leq v$ throughout $M \times [0, T]$. Furthermore, if $u(x_0, t_0) = v(x_0, t_0)$ at some $(x_0, t_0) \in (M \setminus \partial M) \times (0, T]$, then $u \equiv v$ at $M \times (0, t_0]$.

3 Parabolic PDEs on Vector Bundles

In differential geometry, other than the points on the manifold, the objects that we generally work on are vector bundles of the manifold. Instead of just functions on the manifold, we might be interested in the evolution of vector bundles on the manifold, for example the metric or curvature tensors.

Therefore, we need an analogous notion of evolution equation on these vector bundles. Consider a vector bundle $\pi : \mathcal{E} \to M$ and consider the evolution equation for $u : [0, T) \times M \to \mathcal{E}$ given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = L(u) \quad \text{in} \quad M \times (0, T] & (23) \\ u(M, 0) = u_0(M) & (24) \end{cases}$$

where $L : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ is some differential operator map. If the operator is a bundle homomorphism, then this equation is linear.

An example of this type of equation is the Ricci flow. In the case of Ricci flow, the quantity u in equation (23) is g, which is a section of the symmetric positive definite bundle $\mathrm{Sym}^+(T^*M \otimes T^*M)$ and the map $L : C^\infty(\mathrm{Sym}^+(T^*M \otimes T^*M)) \to C^\infty(\mathrm{Sym}(T^*M \otimes T^*M))$ is the Ricci tensor which sends a metric g to a symmetric 2-tensor. However, this map is not a bundle homomorphism, so the Ricci flow is not a linear PDE. As in the previous section, we can linearise this PDE to investigate its behaviour, which we will do in later sections!

3.1 Symbol and Principle Symbol of Linear PDEs

Consider two vector bundles $\pi : \mathcal{E} \to M$ and $\tau : \mathcal{F} \to M$. We consider first the case where L is an arbitrary linear differential operator of order k mapping $C^\infty(\mathcal{E})$ to $C^\infty(\mathcal{F})$, which can be written as:

$$L(u) = \sum_{|\alpha| \leq k} L^\alpha \partial^\alpha u$$

where $L^\alpha \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$, are bundle homomorphisms (fibrewise linear maps). Similar to the linear scalar PDEs, we can define the symbol $\sigma_L(x, \xi)$ of the operator L as a map $\sigma_L : T^*M \to \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ given by:

$$\sigma_L(x, \xi) = \sum_{|\alpha| \leq k} L^\alpha \xi_\alpha$$

where (x, ξ) is a local trivialisation of the cotangent bundle. This definition respects composition because homomorphism respects composition. Suppose that K is another linear differential operator over the vector bundle such that $M : C^\infty(\mathcal{F}) \to C^\infty(\mathcal{G})$, then the symbol $\sigma_{M \circ L} : T^*M \to \mathrm{Hom}(\mathcal{E}, \mathcal{G})$ with the property:

$$\sigma_{M \circ L}(x, \xi) = \sigma_M(x, \xi) \circ \sigma_L(x, \xi)$$

where σ_L and σ_M are symbols for the differential operators L and M respectively.

Again, similar to the scalar case, we are mostly interested in the leading term of the symbol $\hat{\sigma}_L(x, \xi)$, which is again called the principal symbol, given by:

$$\hat{\sigma}_L(x, \xi) = \sum_{|\alpha| = k} L^\alpha \xi_\alpha$$

Principal symbols are important in the theory of PDEs on vector bundles because it allows us to determine the type of the PDE.

Remark 3.1. For (23)-(24) to make sense, then necessarily we have $\mathcal{E} = \mathcal{F}$ by inspecting both ides of the equation. Note that in the Ricci flow equation $\mathcal{E} = \mathrm{Sym}^+(T^*M \otimes T^*M)$ and $\mathcal{F} = \mathrm{Sym}(T^*M \otimes T^*M)$, so everything is good so far.

Remark 3.2. Some literature, for example [Top], a different definition for the principal symbol of the operator L is sometimes used, which is independent of coordinates. Given $(x, \xi) \in T^*M$ and $v \in C^\infty(\mathcal{E})$ and $\phi : M \to \mathbb{R}$ with $d\phi(x) = \xi$, the principal symbol of the operator L is defined as:

$$\hat{\sigma}_L(x, \xi)v = \lim_{s \to \infty} s^{-2} e^{-s\phi(x)} L(e^{s\phi(x)} v)(x)$$

3.2 Linear Second Order Parabolic PDEs

We let M be a closed oriented manifold equipped with the metric $g(t)$ and $\pi : \mathcal{E} \to M$ is a vector bundle over M with bundle metric h. We define the connection on $C^\infty(\mathcal{E})$ which is compatible with the metric h by $D^{\mathcal{E}}$. Now we want to define a second covariant derivative on this vector bundle. Thus, we define a connection $D^{\mathcal{E}\otimes T^*M}$ on the tensor $C^\infty(\mathcal{E}\otimes T^*M)$ such that for all $X \in TM, \xi \in T^*M$ and $\varphi \in C^\infty(\mathcal{E})$, we have

$$D_X^{\mathcal{E}\otimes T^*M}(\varphi\otimes\xi) = (D_X^{\mathcal{E}}\varphi)\otimes\xi + \varphi\otimes(\nabla_X^{T^*M}\xi)$$

The second covariant derivatives is then defined as the composition of the connection on $C^\infty(\mathcal{E})$ and the connection on $C^\infty(\mathcal{E}\otimes T^*M)$.

$$C^\infty(\mathcal{E}) \xrightarrow{D^{\mathcal{E}}} C^\infty(\mathcal{E}\otimes T^*M) \xrightarrow{D^{\mathcal{E}\otimes T^*M}} C^\infty(\mathcal{E}\otimes T^*M\otimes T^*M)$$

Definition 3.1 (Second covariant derivative)**.** We define the second covariant derivative on a vector bundle E as the map $D^2 : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E}\otimes T^*M\otimes T^*M)$ given by for any $X, Y \in TM$:

$$(D^2\varphi)(X,Y) = D^2_{XY}\varphi = D_X^{\mathcal{E}}(D_Y^{\mathcal{E}}\varphi) - D^{\mathcal{E}}_{\nabla_X Y}\varphi$$

where ∇ is the Levi-Civita connection on TM compatible with $g(t)$. This expression is $C^\infty(M)$-linear over X and Y.

Then, we can define the Laplacian on the vector bundle $C^\infty(\mathcal{E})$ by taking the negative of the trace over $g(t)$ of the second covariant derivatives on φ, as:

$$\Delta_g\varphi = -\mathrm{tr}_{g(t)}(D^2\varphi) = -\sum_{i,j=1}^{n} g(t)^{ij}(D^2\varphi)(\partial_i,\partial_j)$$

Having a well defined analogue of a Laplacian for vector bundle allows us to write down a parabolic type equation like in (8)-(9). Consider the equation (23)-(24) for $u : M\times[0,T) \to \mathcal{E}$ where the operator L is a linear operator of the form

$$L(u) = -\Delta_g u + D^{\mathcal{E}}_{X(t)}u + c(x,t)u \tag{25}$$

for some smooth vector field $X(t)$ and $c(x,t) \in \mathrm{Hom}(\mathcal{E},\mathcal{E})$. In terms of local coordinates $\{x^i\}$ of M, picking a local frame $\{e_\alpha\}$ of $\mathcal{E}$, we can write the operator L as

$$L(u) = \left(a^{ij}_{\alpha\beta}\frac{\partial^2 u^\beta}{\partial x^i\partial x^j} + b^i_{\alpha\beta}\frac{\partial u^\beta}{\partial x^i} + c_{\alpha\beta}u^\beta\right)e_\alpha$$

Thus, comparing both sides, equation (23) is a component-wise system of PDEs. As in the previous section, we want to classify a type of PDE which is called the parabolic

PDE. These type of PDEs enjoy some nice properties which will be discussed later. In order to classify the PDE, we look at the principal symbol of the equation.

Definition 3.2. The partial differential equation in (23) on a Riemannian manifold (M, g) is parabolic if the principal symbol of L, $\hat{\sigma}_L(x, \xi)$ is a bundle isomorphism between $\mathcal{E}$ and $\mathcal{F}$ for ever non-zero $\xi \in T^*M \setminus \{0\}$. This is equivalent to there exists $\lambda > 0$ such that

$$h(\hat{\sigma}_L(x, \xi)v, v) \geq \lambda|\xi||v|$$

where h is any bundle metric on $\mathcal{E}$.

Remark 3.3. Again like in the previous section, we may have the notion of weak and strong parabolicity. Weak parabolicity occurs when the principal symbol is allowed to not be an isomorphism whenever $\xi \in T^*M \setminus \{0\}$. We shall see later that this is an important distinction because the Ricci flow equation, after linearisation, is not even strongly parabolic. But, it can be fixed to be a strongly parabolic type!

3.3 Non-Linear Second Order Parabolic PDEs

As in the previous section, suppose that $\tilde{F}(x, u)$ is a non-linear operator on a vector bundle $\mathcal{E}$ involving the terms u and all its derivatives. This is a more realistic form of a PDE which crops up more frequently in nature. We are going to linearise this operator and investigate the linear behaviour of the second order equation

$$\frac{\partial u}{\partial t} = \tilde{F}(x, u) = F(x, t, u, \partial_x u, \partial_x^2 u) \tag{26}$$

Suppose that $v : [0, T) \to C^\infty(\mathcal{E})$ is a time-dependent smooth section of the vector bundle over a closed manifold M. We define the linearisation of the map $\tilde{F}$ at u to be the linear map

$$L(v) = (D\tilde{F})(v) = \frac{d}{d\varepsilon}\tilde{F}(x, u + \varepsilon v)|_{\varepsilon=0}. \tag{27}$$

Definition 3.3 (Non-linear second order parabolic PDE)**.** The non-linear equation (26) is called parabolic non-linear equation at u if the linear equation involving the linearisation of F at u i.e. Lv for sufficiently smooth v, given by

$$\frac{\partial v}{\partial t} = L(v) \quad \text{in} \quad M \times (0, T] \tag{28}$$

is parabolic in the sense of Definition 3.2.

3.4 Some Properties of Parabolic PDEs on Vector Bundles

Here we are going to state some theorems that we can deduce from the second order parabolic equations. We begin with a general existence and uniqueness theory.

Theorem 3.1. [Top, p.58] Suppose that the non-linear equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = F(u) & (29) \\ u(M,0) = w(M) & (30) \end{cases}$$

is parabolic at w. Then, there exists an $\varepsilon > 0$ such that the equation has a unique smooth solution of $v \in C^{\infty}(\mathcal{E}) \times [0, \varepsilon)$.

A nice feature of parabolic type equations which is useful in our study is the maximum principles, just like in the scalar case. However, for the scalar case, we have a well-defined notion of order in the set $\mathbb{R}$: the solution at any interior point away from the initial time is controlled by (smaller than or bigger than) the values at the boundary or initial time.

We do not have a similar notion of order in vector bundles. One thing to note is that the parabolic equation evens out the scalar quantity that is evolving. Thus, we expect for vector bundle evolution equation, some property of the solution is controlled at the boundary and the initial time. This property is the convexity property (or sometimes called avoidance property, depending on how the theorem is stated) which states that the solution will remain within a convex set of the vector bundle $\mathcal{E}$ for all time.

Theorem 3.2. [CK, p.101] Assume that $u(t)$ is a solution of the nonlinear PDE

$$\frac{\partial u}{\partial t} = \Delta_g u + f(u)$$

for $t \in [0, T]$ such that $f : \mathcal{E}_x \to \mathcal{E}_x$ is smooth and fibre-preserving, and $u(0) \in \mathcal{K}(0)$ where $\mathcal{K}(0)$ is a closed subset of $\mathcal{E}$. Assume further that

- the space-time track $(\mathcal{K}(t), t)$ is a closed subset of $\mathcal{E} \times [0, T]$.
- $\mathcal{K}(t)$ is invariant under parallel translation by $D^{\mathcal{E}}(t)$ for all $t \in [0, T)$
- $\mathcal{K}_x(t) = \mathcal{K}(t) \cup \pi^{-1}(x)$ is a closed convex subset of $\mathcal{E}_x$ for all $x \in M$ and $t \in [0, T)$.

Then, if every solution of the ODE

$$\begin{cases} \dfrac{du}{dt} = f(u) \\ u(0) \in \mathcal{K}_x \end{cases}$$

is defined in each fibre $\mathcal{E}_x$ remains in $\mathcal{K}_x(t)$ for all t, then $u(t)$ remains in $\mathcal{K}(t)$ for all t.

A special case which is useful in the study of the Ricci flow is the following corollary:

Corollary 3.1. Let $g(t)$ be a smooth family of Riemannian metrics on a closed manifold M. Let $u(t) \in C^\infty(\mathrm{Sym}(T^*M \otimes T^*M))$ be a symmetric 2-tensor satisfying the semilinear equation

$$\frac{\partial u}{\partial t} = \Delta_{g(t)} u + f(u, g, t)$$

where $f(u, g, t)$ is a symmetric 2-tensor which is locally Lipschitz in all its arguments and satisfies the null-eigenvector assumption, that is $f(x,t)(v,v) \geq 0$ for every null-eigenvector of $u(x,t)$ i.e. $u(x,t)v = 0$. If $u(0)$ is positive semi-definite, then $u(t)$ is positive semi-definite for all t whenever the solution exists.

The above corollary will ensure that the solution of the Ricci flow, as long as it exists, is still a positive definite 2-tensor and thus will remain a metric.

4 Ricci Flow

Now, we prove equation (3) that we claimed in the introductory section. Harmonic coordinates of an n-dimensional manifold M is a set of local coordinates $\{x^i\}_{i=1}^n$ such that $\Delta x^i = 0$ for all $i = 1, \dots n$ where Δ is the Laplace-Beltrami operator. As a consequence of this choice of coordinate, we have that $g^{ij}\Gamma_{ij}^k = 0$ for all $k = 1, \dots n$. Using the formula

$$g^{ij}\Gamma_{ij}^k = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^r} (\sqrt{|g|} g^{rk})$$

implies that the lower order terms in Δg_{ij} vanishes, thus Δg_{ij} is simply:

$$\Delta g_{ij} = -g^{ij} \frac{\partial^2 g_{ij}}{\partial x^i \partial x^j}$$

In local coordinates, by expanding the terms and collecting the quadratic terms in g^{-1} and ∂g, the Ricci tensor R_{ij} can be written locally as:

$$\begin{aligned}
R_{ij} &= \frac{\partial \Gamma_{ij}^p}{\partial x^p} - \frac{\partial \Gamma_{ip}^p}{\partial x^j} + \Gamma_{ij}^p \Gamma_{pq}^q - \Gamma_{ip}^q \Gamma_{jq}^p \\
&= \frac{1}{2} g^{pr} \left(\frac{\partial}{\partial x^p} \left(\frac{\partial g_{ri}}{\partial x^j} + \frac{\partial g_{rj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^r} \right) - \frac{\partial}{\partial x^j} \left(\frac{\partial g_{ri}}{\partial x^p} + \frac{\partial g_{rp}}{\partial x^i} - \frac{\partial g_{ip}}{\partial x^r} \right) \right) + Q_{ij}(g^{-1}, \partial g) \\
&= \frac{1}{2} \left(\Delta g_{ij} + g^{pr} \left(\frac{\partial^2 g_{rj}}{\partial x^p \partial x^i} + \frac{\partial^2 g_{ip}}{\partial x^j \partial x^r} - \frac{\partial^2 g_{rp}}{\partial x^i \partial x^j} \right) \right) + Q_{ij}(g^{-1}, \partial g) \\
&= \frac{1}{2} \left(\Delta g_{ij} + g_{il} \frac{\partial}{\partial x^j} (\cancel{g^{pr}\Gamma_{pr}^l}) + g_{jl} \frac{\partial}{\partial x^i} (\cancel{g^{pr}\Gamma_{pr}^l}) \right) + Q_{ij}(g^{-1}, \partial g) \\
&= \frac{1}{2} \Delta g_{ij} + Q_{ij}(g^{-1}, \partial g)
\end{aligned}$$

where $Q_{ij}(g^{-1}, \partial g)$ is the collected quadratic terms in g^{-1} and derivatives of g. This proves (3). Thus, comparing with the heat equation in (10), the equation for the Ricci flow (1) is indeed of the form of a non-linear heat equation, due to the presence of the term $Q_{ij}(g^{-1}, \partial g)$.

4.1 Ricci Flow as a Weakly Parabolic Equation

However, this only gives us a formal intuition of the behaviour of the equation because in this coordinates, we have one big problem. We cannot guarantee that the chosen harmonic coordinates at the initial metric will be still be harmonic after the metric is flowed for even a small amount of time. In fact, this formula can be quite misleading: the Ricci flow is not strongly parabolic!

To see this, we linearise the Ricci curvature tensor in the sense of Definition 19. We define the linearisation of the Ricci tensor as (27) by picking a positive symmetric 2-tensor h_{ij} so that we have the following lemma.

Lemma 4.1. The linearisation of the Ricci tensor, is given by

$$(D\mathrm{Ric})(h)_{ij} = \frac{1}{2} g^{pq}(\nabla_p \nabla_i h_{qj} + \nabla_q \nabla_j h_{ip} - \nabla_p \nabla_q h_{ij} - \nabla_i \nabla_j h_{pq}) + \cdots \tag{31}$$

Proof. Define the Christoffel symbols for the metric $g + \varepsilon h$ as $\hat{\Gamma}$. This makes sense as the space of Riemannian metric on a compact manifold forms an open convex cone. In coordinates, we have

$$\begin{aligned}
(D\mathrm{Ric})(h)_{ij} &= \frac{\partial}{\partial \varepsilon} \mathrm{Ric}(g_{ij} + \varepsilon h_{ij})|_{\varepsilon=0} \\
&= \frac{\partial}{\partial \varepsilon} \left(\frac{\partial \hat{\Gamma}^p_{ij}}{\partial x^p} - \frac{\partial \hat{\Gamma}^p_{ip}}{\partial x^j} + \hat{\Gamma}^q_{ij} \hat{\Gamma}^p_{pq} - \hat{\Gamma}^q_{ip} \hat{\Gamma}^p_{jq} \right) \Bigg|_{\varepsilon=0} \\
&= \frac{1}{2} g^{pq} \frac{\partial}{\partial x^p} \left(\frac{\partial h_{qi}}{\partial x^j} + \frac{\partial h_{qj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^q} \right) - \frac{1}{2} g^{pq} \frac{\partial}{\partial x^j} \left(\frac{\partial h_{qi}}{\partial x^p} + \frac{\partial h_{pq}}{\partial x^i} - \frac{\partial h_{ip}}{\partial x^q} \right) + \cdots \\
&= \frac{1}{2} g^{pq}(\nabla_p \nabla_i h_{qj} + \nabla_q \nabla_j h_{ip} - \nabla_p \nabla_q h_{ij} - \nabla_i \nabla_j h_{pq}) + \cdots
\end{aligned}$$

which is what we wanted to proof. □

To check the parabolicity of the Ricci flow, we calculate the principal symbol of the non-linear operator $-2\mathrm{Ric}(g(t))$. For a covector $\xi \in T^*M$, the principal symbol is given by

$$(\hat{\sigma}_{-2\mathrm{Ric}}(x, \xi) h)_{ij} = g^{pq}(\xi_p \xi_q h_{ij} + \xi_i \xi_j h_{pq} - \xi_p \xi_i h_{qj} - \xi_q \xi_j h_{ip}) \tag{32}$$

The linearisation of the operator at g is parabolic if there exists a positive function $\lambda(x) > 0$ such that for all positive definite $h \in \mathrm{Sym}^+(T^*M \otimes T^*M)$ and non-zero $\xi \in T^*M$, we have

$$\langle \hat{\sigma}_{-2\mathrm{Ric}}(x, \xi)h, h\rangle \geq \lambda(x)|\xi|^2|h|^2$$

where $\langle \cdot, \cdot \rangle$ is an induced inner product on the vector bundle $\mathrm{Sym}(T^*M \otimes T^*M)$.

However, if we choose $h_{ij} = \xi_i \xi_j$ (so h is symmetric and positive definite), we can see that the LHS vanishes. Therefore, the Ricci flow is not strongly parabolic. It is only weakly parabolic because the value 0 is attained by some ξ and h.

This is somehow predictable: if we assume that the equation is parabolic, then the stationary solution of the Ricci flow solves an elliptic equation, namely $\mathrm{Ric}(g(t)) = 0$ for all $t > 0$. We know that the space of solutions of elliptic equation on a compact manifold is finite dimensional, so we expect the solution space for the stationary Ricci flow to be finite dimensional. However, the Ricci tensor is invariant under the full diffeomorphism group (which is infinite dimensional). So, given a stationary solution of the Ricci flow, we can construct another linearly independent solution by pulling it back by any diffeomorphism, which gives us a contradiction!

Since the Ricci flow is only weakly parabolic, we cannot utilise the existence theory we had in the previous sections as they require the equation to be strongly parabolic. However, Hamilton managed to show prove a the short-time existence of the Ricci flow if the manifold is compact and the initial metric is smooth using the Nash-Moser inverse function theorem, which is quite technical. Later, DeTurck managed to give a short-time existence result in a much simpler way which is today known as the DeTurck's Trick. This is what we are going to look at in the next section.

4.2 DeTurck's Trick

DeTurck's trick enables us to express the weakly parabolic Ricci flow in a strongly parabolic manner. Thus, this allows us to exploit the theories we have developed for parabolic equations in previous sections. The crux of the trick is that, as we noted before, the Ricci tensor is invariant under the full diffeomorphism group.

At the initial time, we can find a coordinate system that makes the Ricci flow strongly parabolic, which is the harmonic coordinates. To keep the flow strongly parabolic at all times, we can try and construct a family of time dependent diffeomorphism of the manifold onto itself, ϕ_t which evolves along with the Ricci flow, that preserves the parabolicity of the equation. By pulling the solution of this modified flow back by ϕ_t, we get our original Ricci flow.

The equation for this modified flow is often called the DeTurck-Ricci flow and it is given as:

$$\begin{cases} \dfrac{\partial}{\partial t}\bar{g}(t) = -2\mathrm{Ric}(\bar{g}(t)) + \mathcal{L}_{X(t)}\bar{g}(t) & (33) \\ \bar{g}(0) = \bar{g}_0 & (34) \end{cases}$$

where $\mathcal{L}_{X(t)}$ is the Lie derivative in the direction of $X(t)$, which is the vector field generating the diffeomorphism ϕ_t such that $\phi_0 = \mathrm{id}$.

If $\bar{g}$ solves the DeTurck-Ricci flow (33)-(34), then it can be shown that the pullback of this metric by the diffeomorphisms ϕ_t, namely $g(t) = \phi_t^*\bar{g}(t)$, solves the Ricci flow (1)-(2) with $g_0 = \bar{g}_0$. See [AH] for more details and proof of this. Thus, the two problems are equivalent, despite the fact that the DeTurck-Ricci flow has the advantage of being strongly parabolic and enjoys the parabolic theory we have developed earlier.

First, we are going to inspect the linearisation of the Ricci tensor (31) and see which terms are the bad terms which make the equation weakly parabolic. By symmetry of the metric tensor g, we can write the linearisation of the tensor $-2\mathrm{Ric}(g(t))$ as:

$$(D(-2\mathrm{Ric}))(h)_{ij} = g^{pq}(\nabla_p\nabla_q h_{ij} + \nabla_i\nabla_j h_{pq} - \nabla_q\nabla_i h_{jp} - \nabla_q\nabla_j h_{ip}) + \cdots \tag{35}$$

To make the expression simpler, we define a covector V on M by the following:

$$V = g^{pq}\left(\frac{1}{2}\nabla_i h_{pq} - \nabla_q h_{pi}\right) dx^i = \frac{1}{2}g^{pq}(\nabla_i h_{pq} - \nabla_q h_{pi} - \nabla_p h_{qi})\, dx^i.$$

Remark 4.1. The expression for V_i is actually $-g^{pq}g_{ir}(D(\Gamma)(h))^r_{pq}$, which is useful later.

Since $\nabla g = 0$, we can write (35) as:

$$(D(-2\mathrm{Ric}))(h)_{ij} = g^{pq}\nabla_p\nabla_q h_{ij} + \nabla_i V_j + \nabla_j V_i + \cdots \tag{36}$$

The first term in the expression above is a good term, because it is essentially the Laplacian (or the Laplace-Beltrami operator on 0-forms), which is parabolic as we have seen in Example 2.1. The remaining second order terms are the bad terms: we aim to get rid of these terms by introducing a cancelling time-dependent diffeomorphism.

We define a new metric $\bar{g}$ such that $g(t) = \phi_t^*\bar{g}(t)$ for some time-dependent diffeomorphism ϕ_t. We are going to construct a parabolic equation for $\bar{g}(t)$. Since $g(t)$ satisfies the Ricci flow and the Ricci tensor is diffeomorphically invariant, we have the following:

$$\begin{aligned} \frac{\partial}{\partial t}(\phi_t^*\bar{g}(t)) &= \frac{\partial}{\partial t}g(t) \\ \Rightarrow \qquad \phi_t^*\left(\frac{\partial}{\partial t}\bar{g}(t)\right) + \phi_t^*(\mathcal{L}_{X(t)}\bar{g}(t)) &= -2\mathrm{Ric}(g(t)) \\ &= -2\mathrm{Ric}(\phi_t^*\bar{g}(t)) = -2\phi_t^*\mathrm{Ric}(\bar{g}(t)) \end{aligned}$$

where $X(t)$ is a time-dependent vector field defined by:

$$\begin{cases} \dfrac{\partial}{\partial t}\phi_t = X(t) \\ \phi_0 = \mathrm{id} \end{cases}$$

Thus, the metric $\bar{g}(t)$ satisfies the equation

$$\frac{\partial}{\partial t}\bar{g}(t) = -2\mathrm{Ric}(\bar{g}(t)) - \mathcal{L}_{X(t)}\bar{g}(t) := P(\bar{g}(t)) \tag{37}$$

If we linearise the operator $P(\bar{g}(t))$, we would get a similar term to (36) with an extra term coming from the linearisation of $\mathcal{L}_{X(t)}\bar{g}(t)$, namely:

$$(DP)(h)_{ij} = \bar{g}^{pq}\nabla_p\nabla_q h_{ij} + \nabla_i V_j + \nabla_j V_i - D(\mathcal{L}_{X(t)}\bar{g}(t))(h) + \cdots \tag{38}$$

Now, we want to define the vector field $X(t)$ such that the linearisation of $\mathcal{L}_{X(t)}\bar{g}(t)$ is equal to the bad term in (36). In fact, we can explicitly find a suitable vector field $X \in C^\infty(TM)$ to make the equation parabolic. We note that the linearisation of the Christoffel symbols of $\bar{g}(t)$ is given by:

$$D(\bar{\Gamma})(h)^k_{ij} = \frac{1}{2}\bar{g}^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}) + \cdots$$

which almost looks like V except for the index k. Therefore, a suitable choice for the vector field X such that its related one form is given by:

$$X^\flat = -\bar{g}_{ij}\bar{g}^{pq}(\bar{\Gamma}^j_{pq} - \Xi^j_{pq})dx^i$$

where $\bar{\Gamma}^k_{ij}$ is the Christoffel symbol of the metric $\bar{g}$ and Ξ^k_{ij} is the Christoffel symbol of some fixed background metric on the manifold. The background metric is required to ensure that this vector field transforms tensorially.

This choice for X ensures that the bad term disappears, making the flow (33)-(34) strongly parabolic. Thus, by parabolic theory, a unique solution $\bar{g}(t)$ exists for some small time interval. Pulling back the this metric by the diffeomorphism ϕ_t generated by this choice of X gives us a solution for the Ricci flow (1)-(2). Furthermore, this solution is unique. This can be shown by using a different type of flow called the harmonic map flow. See [CK] for details.

References

[AH] Andrews, B. and Hopper, C. *The Ricci Flow in Riemannian Geometry: A Complete Proof of the Differentiable 1/4-Pinching Theorem*. Springer-Verlag, Berlin (2011).

[Aub] Aubin, T. *Some Non-linear Problems in Riemannian Geometry*. Springer, Berlin (1991).

[CK] Chow, B. and Knopf, D. *The Ricci Flow: An Introduction*. American Mathematical Society, Providence (2004).

[Eva] Evans, L.C. *Partial Differential Equations*. American Mathematical Society, Providence (2002).

[Ham] Hamilton, R.S. *Harmonic Maps of Manifolds with Boundary*. Springer-Verlag, Berlin (1975).

[Kaz] Kazdan, J.L. *Applications of Partial Differential Equations to Problems in Geometry*. Available from: `https://www.math.upenn.edu/~kazdan/`

[Top] Topping, P. *Lectures on the Ricci Flow*. Cambridge University Press, Cambridge (2006).